LECTURE-6

Linear DE

The general form of linear DE is

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \frac{dx}{dy} + P(y)x = Q(y)$$

Ex:-

$$\frac{dy}{dx} + e^x y = \sin x$$

Ex:-

$$\frac{dx}{dy} + y^2 x = \cos y$$

Ex $\frac{dy}{dx} + y = 1$

Ex $\frac{dx}{dy} + e^y x = \sinh y$

Ex $x\frac{dy}{dx} + y = \sin x$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x}$$

How to solve ?

Let us consider the linear DE

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{---(1)}$$

STEP:-I

The integrating factor is $e^{\int P(x)\,dx}$

STEP:-II

Multiplying the IF into both sides of equation (1) term by term

$$\frac{dy}{dx} e^{\int P(x)\,dx} + P(x)y\, e^{\int P(x)\,dx} = Q(x)\, e^{\int P(x)\,dx}$$

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$$\Rightarrow \frac{d}{dx}\left(y\, e^{\int P(x)\,dx} \right) = Q(x)\, e^{\int P(x)\,dx}$$

D.V x I.F

$$\Rightarrow d\left(y\, e^{\int P(x)\,dx} \right) = Q(x)\, e^{\int P(x)\,dx}\, dx$$

Integrating both sides

$$\int d\left(y\, e^{\int P(x)\,dx} \right) = \int Q(x)\, e^{\int P(x)\,dx}\, dx + C.$$

NOTE:-

Similarly if we consider the linear DE $\frac{dx}{dy} + P(y)\,x = Q(y)$

Interchanging x & y in the previous result we can get the solution

$$\int d\left(x\, e^{\int P(y)\,dy} \right) = \int Q(y)\, e^{\int P(y)\,dy}\, dy + C$$

~~Example~~

PAGE -38, E. KREYSZIG (8TH EDITION)

3-TO-14

Solve the following

7. $y' + Ky = e^{-Kx}$

$\Rightarrow \frac{dy}{dx} + Ky = e^{-Kx}$ which is a linear DE with

(1)

$P(x) = K$, $Q(x) = e^{-Kx}$

The integrating factor is $e^{\int P(x)\,dx} = e^{\int K\,dx} = e^{Kx}$

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Multiplying the I.F into both sides of equation (1)

$$\frac{dy}{dx} e^{kx} + ke^{kx} y = e^{-kx} e^{kx}$$

$$\Rightarrow \frac{d}{dx}(ye^{kx}) = 1$$

$$\Rightarrow d(ye^{kx}) = 1\,dx$$

Integrating both sides

$$\int d(ye^{kx}) = \int dx$$

$$\Rightarrow ye^{kx} = x + C.$$

15 - TO - 22

Solve the following initial value problem

22. $xy' + 4y = 8x^4 \qquad y(1) = 2$

$$\Rightarrow x\frac{dy}{dx} + 4y = 8x^4$$

$$\Rightarrow \frac{dy}{dx} + \frac{4}{x} y = 8x^3 \quad \text{——— (1)}$$

which is a linear DE with $P(x) = \frac{4}{x}$, $Q(x) = 8x^3$

The integrating factor is

$$e^{\int P(x)\,dx} = e^{\int \frac{4}{x} dx} = e^{4\ln|x|} = e^{\ln|x^4|} = x^4$$

Multiplying the I.F into both sides of equation (1)

$$x^4 \frac{dy}{dx} + 4x^3 y = 8x^7$$

$$\Rightarrow \frac{d}{dx}(yx^4) = 8x^7$$

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$\Rightarrow d(yx^4) = 8x^7\,dx$

Integrating both sides,

$$\int d(yx^4) = \int 8x^7\,dx$$

$$\Rightarrow yx^4 = x^8 + C \quad \text{——} \quad (2)$$

$y(1) = 2$ i.e $y = 2$ when $x = 1$ from equ^n (2)

$$2 \cdot 1^4 = 1^8 + C \Rightarrow C = 1$$

Putting the value of C in equation (2) the solution is

$$yx^4 = x^8 + 1$$

16. $y' - (1 + 3x^{-1})y = x + 2 \qquad y(1) = e - 1$

$$\Rightarrow \frac{dy}{dx} - \left(1 + \frac{3}{x}\right)y = x + 2 \quad \text{——} \quad (1)$$

which is a linear DE with $P(x) = -\left(1 + \frac{3}{x}\right)$

$$Q(x) = x + 2$$

So, the I.F $= e^{\int -\left(1 + \frac{3}{x}\right)dx}$

$$= e^{-x - 3\ln|x|}$$

$$= e^{-x} \cdot e^{\ln|x^{-3}|}$$

$$= e^{-x}\, x^{-3} = \frac{e^{-x}}{x^3}.$$

Multiplying the I.F into both sides of equation (1)

$$\frac{dy}{dx}\,\frac{e^{-x}}{x^3} - \left(1 + \frac{3}{x}\right)\frac{e^{-x}}{x^3}\,y = (x+2)\,\frac{e^{-x}}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y\,e^{-x}}{x^3}\right) = \frac{e^{-x}}{x^2} + \frac{2e^{-x}}{x^3}$$

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$$\Rightarrow d\left(\frac{ye^{-x}}{x^3}\right) = \frac{e^{-x}}{x^2}dx + \frac{2e^{-x}}{x^3}dx$$

Integrating both sides,

$$\int d\left(\frac{ye^{-x}}{x^3}\right) = \int \frac{e^{-x}}{x^2}dx + \int \frac{2e^{-x}}{x^3}dx$$

$$\Rightarrow \int d\left(\frac{ye^{-x}}{x^3}\right) = -\frac{e^{-x}}{x^2} - \int -e^{-x}\cdot -2x^{-3}dx + \int \frac{2e^{-x}}{x^3}dx$$

$$\Rightarrow \frac{ye^{-x}}{x^3} = -\frac{e^{-x}}{x^2} - \int \frac{2e^{-x}}{x^3}dx + \int \frac{2e^{-x}}{x^3}dx + C$$

$$\Rightarrow \frac{ye^{-x}}{x^3} = -\frac{e^{-x}}{x^3} + C \quad \text{---(2)}$$

$$\Rightarrow \frac{y}{x^3} = -\frac{1}{x^2} + ce^{x}$$

$$\Rightarrow y = -x + cx^3e^{x} \quad \text{---(2)}$$

$y(1) = e-1$ i.e $y = e-1$ when $x=1$ from (2)

$$e-1 = -1 + ce \Rightarrow e = ce$$

$$\Rightarrow c = 1$$

Putting the value of c in equation (2) the solution is

$$y = -x + x^3e^{x}.$$

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34, 35

Solve the following

35. $y' = \dfrac{1}{(6e^y - 2x)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{6e^y - 2x}$

$\Rightarrow \dfrac{dx}{dy} = 6e^y - 2x$

$\Rightarrow \dfrac{dx}{dy} + 2x = 6e^y$ ——— (1)

which is a Linear DE with $P(y) = 2$, $Q(y) = 6e^y$

The integrating factor is $e^{\int P(y)dy} = e^{\int 2dy} = e^{2y}$

Multiplying the I.F into both sides of equation (1)

$\dfrac{dx}{dy} e^{2y} + 2e^{2y} x = 6e^y e^{2y}$

$\Rightarrow \dfrac{d}{dy}(xe^{2y}) = 6e^{3y}$

$\Rightarrow d(xe^{2y}) = 6e^{3y} dy$

Integrating both sides.

$\int d(xe^{2y}) = \int 6e^{3y} dy$

$\Rightarrow xe^{2y} = 6\dfrac{e^{3y}}{3} + C$

$\Rightarrow xe^{2y} = 2e^{3y} + C$.

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